Teaching Discrete Mathematics to Early Undergraduates with Software Foundations

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spoiler alert

- **early undergrads** can use Coq (**CS2**)
- they learn **real discrete mathematics**
- **informal** and **formal** proof **synergize**
- Coq demands **careful logistics**

my plan for the talk

* background
* pedagogical idea
* what we taught
* evaluation

Pomona College

1887

* Consortium
  (CMC, HMC, Scripps, Pitzer)

* 5000 students total
  1600 students at Pomona

* Majority minority
  no-loan policy
  12% Pell grants
  17% 1st generation
  3% undocumented

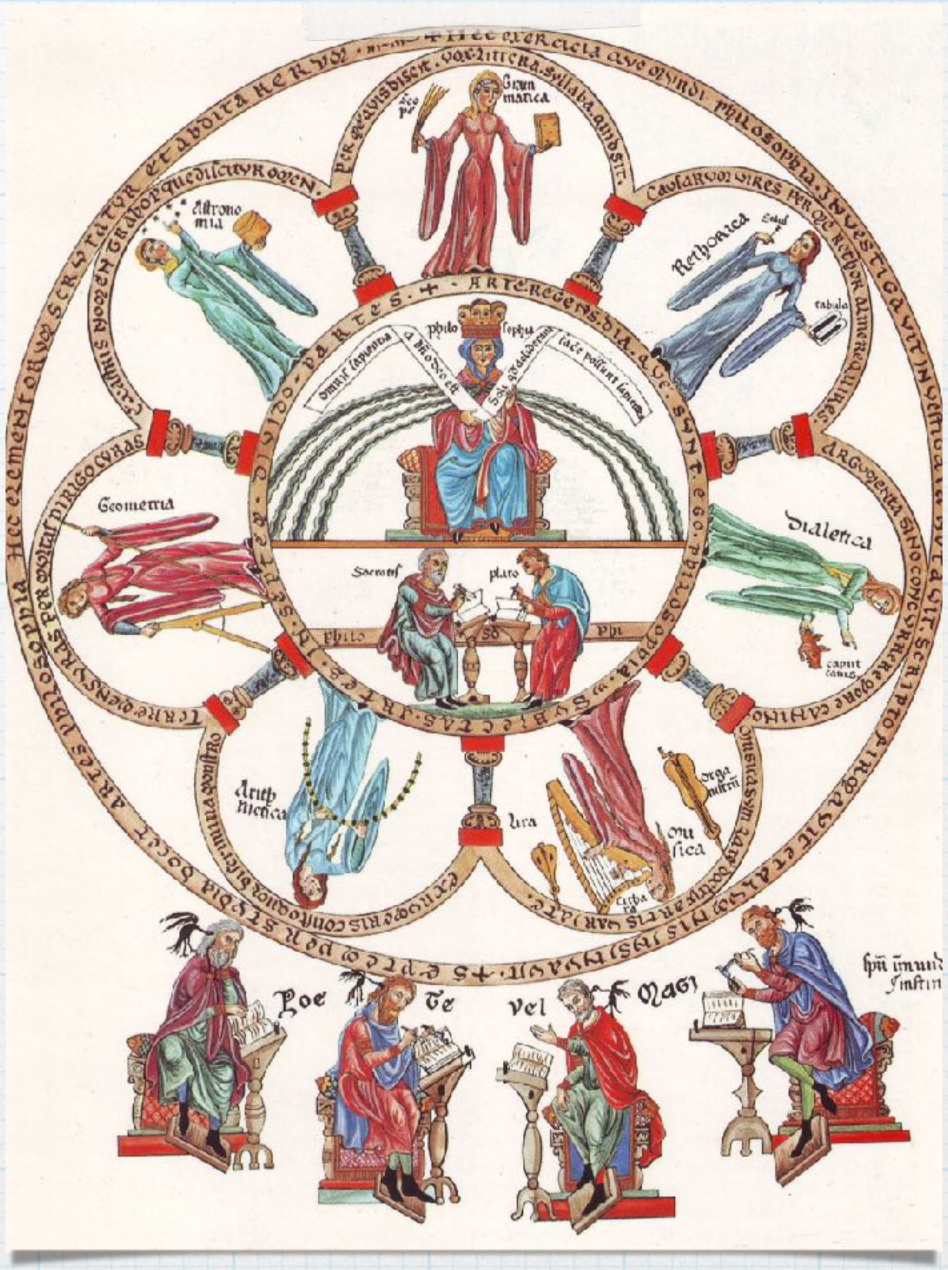
* small classes
  more than 30 is big

septem artes liberales

trivium: grammar, rhetoric, logic

quadrivium: music theory, arithmetic, geometry, astronomy

* students are bright but **may lack background**
* **breadth** over depth
* major is only **11 courses**



existing courses

CS052

* 2nd course in the sequence
* functional programming
* tour of CS topics
    * recursion
    * simple data structures
    * automata
    * Bayesian reasoning

CS055

* pre-req of 4th course (theory)
* discrete math grab bag
    * induction
    * number theory
    * combinatorics
    * probability
    * graph theory

CS054

* functional programming
* proof
* discrete math

CS054 goals

* prove theorems by induction
    * over $\mathbb{N}$ and other structures
* translate between English and propositions
    * first-order logic with sets and inductive propositions
* apply basic graph-theoretic terminology
* program with inductively defined datatypes
    * lists and binary trees

CS054 non-goals

* become idiomatic Coq users
* understand the Curry-Howard Correspondence

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pedagogical idea

the **hard part** of learning proof:
students don't know
the rules of the game

* Coq **enforces the rules...**
* ...until students **internalize them**

based
on a
true story!

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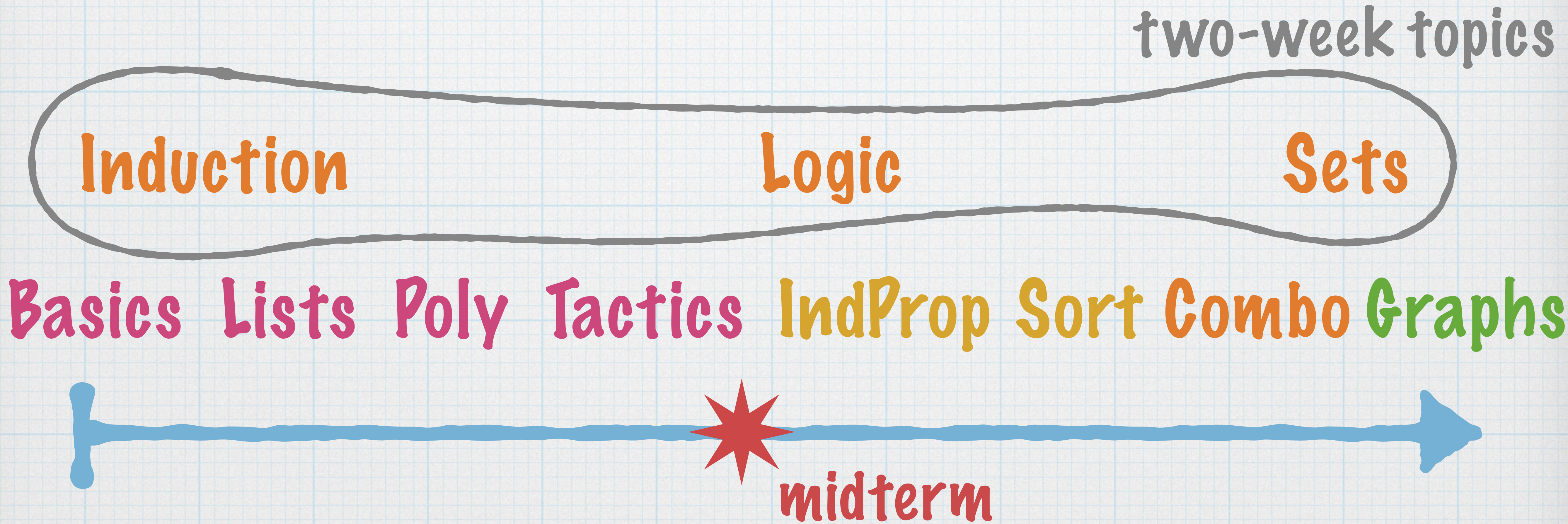
CS054

* Basics, Induction, Lists, Poly, Tactics, Logic, IndProp
  Logical Foundations
  Vol. 1 of Software Foundations (SF)
* Sort
  Appel's Verifying Functional Algorithm's
  Vol. 3 of SF
* Combo, Sets, Graphs
  new!

* lectures mix Coq and chalkboard
* homework in Coq files
  formal and informal proofs
* worksheets not for credit

legend

all Coq
mostly Coq
even mix
all paper



topics over 15 week semester

worksheets

transferring formal to informal

5. Suppose we want to prove that $\forall nm,\ n + Sm = S(n+m)$.

   (a) What can we do induction on? ___ n, m ___

   (b) For each possibility above, list (a) the goal you would have to prove in the base case, (b) the induction hypothesis you would get, and (c) the goal you would have to prove in the induction case.

   > **Solution: Answer for n:**
   > Base case: $0 + Sm = S(0+m)$
   > IH: $n' + Sm = S(n'+m)$
   > Inductive case: $Sn' + Sm = S(Sn'+m)$
   >
   > **Answer for m:**
   > Base case: $n + 1 = S(n+0)$
   > IH: $n + Sm' = S(n+m')$
   > Inductive case: $n + S(Sm') = S(n+Sm')$

   (c) Which of these inductions would work to prove the theorem? ___ n ___

homeworks

Exercise: 3 stars (permutation_length)

You may need to define and prove an auxiliary lemma in order to see how `everywhere` and `leng` interact.

```
Lemma permutation_length :
  ∀ A (l l':list A) (a:A),
  In l' (permutations l) → length (everywhere a l') = S (length l).
Proof.
  (* FILL IN HERE *) Admitted.
```

☐

Exercise: 2 stars (permutations_length)

We can finally prove the desired result: there are `factorial n` permutations of a list of length `n`.

```
Lemma permutations_length :
  ∀ A (l:list A) n,
    length l = n →
    length (permutations l) = factorial n.
Proof.
  (* FILL IN HERE *) Admitted.
```

☐

homeworks

Exercise: 3 stars (lists_of_bools)

How many lists of booleans of length 2 are there? Write them out.

```
(* FILL IN HERE *)
```

How many lists of booleans of length 3 are there? No need to write them out.

```
(* FILL IN HERE *)
```

Write a theorem characterizing how many many lists of booleans of length `n` are there, for any natural `n`. Your proof should be informal.

```
(* FILL IN HERE *)
```

☐

help with tactics

`intros`

Moves things from the goal to the context. It works on quantified variables:

- **FORM**: `intros x y z`
- **WHEN**: goal looks like `forall a b c, H`
- **EFFECT**: add `x`, `y`, and `z` to the context (bound to `a`, `b`, and `c`, respectively); goal becomes `H`
- **INFORMAL**: "Let x, y, and z be given."

`destruct`

Performs case analysis. Its precise use depends on the inductive type being analyzed. Be certain to use `-`/`+`/`*` to nest your case analyses. Always write an `as` pattern.

- **FORM**: `destruct n as [| n']`
- **WHEN**: `n : nat` is in the context
- **EFFECT**: proofs splits into two cases, where `n=0` and `n=S n'` for some `n'`
- **INFORMAL**: "By cases on `n`. - If `n=0` then... - If `n=S n'`, then..." If you're at the beginning of a proof, don't forget to "let `n` be given". It's often good to say what your goal is in each case.

help with proof

CS054 — How to prove it

Text in black is the "script"—it stays the same every time; text in `monospace` is the corresponding Coq code. Text in red is the rest of proof—
have to figure that part out!

Proposition	Pronunciation	How to prove it	How to use it
$\forall x,\ P(x)$	for all x, $P(x)$	Let x be given. Now prove $P(x)$ for this arbitrary x we know *nothing* about. `intros x`	We have y and know $\forall x, P(x)$; therefore, $P(y)$. `apply .../apply ... in ...`
$\exists x,\ P(x)$	there exists an x such that $P(x)$	Let $x =$ choose some object, y. Now prove $P(y)$ for your choice of y. `exists ...`	We have $\exists x,\ P(x)$, so let y be given such that $P(y)$. `destruct ... as [x Hp]`
$p \Rightarrow q$	p implies q; if p, then q	Suppose p. Now prove q, having assumed p. **You don't have to prove p.** `intros H`	**Use #1:** We have $p \Rightarrow q$; since proof of p, we have q. `apply ... in ...` **Use #2:** We must show q, but we have $p \Rightarrow q$, so it suffices to show p. Now go prove p! `apply ...`
$p \wedge q$	p and q	Prove p. Prove q. `split`	We have $p \wedge q$, i.e., we have both p and q. `destruct ... as [Hp Hq]`
$p \vee q$	p or q	**Proof #1:** To see $p \vee q$, we show p. Prove p. **You don't have to prove q.** `left` **Proof #2:** To see $p \vee q$, we show q. Prove q. **You don't have to prove p.** `right`	We have $p \vee q$. We go by cases. (p) If p holds, then prove whatever your goal was, given p. **Ignore q.** (q) If q holds, then prove whatever your goal was, given q. **Ignore p.** `destruct ... as [Hp | Hq]`
$\neg p$	not p	To show $\neg p$, suppose for a contradiction that p holds. Now find a contradiction, like $0 = 1$ or $q \wedge \neg q$ or $5 < 1$. `intros contra; destruct/inversion`	We have $\neg p$; but proof of p—which is a contradiction. **Now you're done with whatever case you're in!** `exfalso; destruct/inversion`
		Derived forms	
$p \Leftrightarrow q$	p iff q; p if and only if q	We prove each direction separately: ($\Rightarrow$) Suppose p; proof of q. ($\Leftarrow$) Suppose q; proof of p.	**Use #1:** We have $p \Leftrightarrow q$; since proof of p, we have q. **Use #2:** We have $p \Leftrightarrow q$; since proof of q, we have p.
$\forall x,\ P(x) \Rightarrow Q(x)$	for all x such that $P(x)$ holds, $Q(x)$ holds	Let an x be given such that $P(x)$. Prove $Q(x)$, given that $P(x)$ holds.	Choose some y. Since we have $P(y)$, we can conclude $Q(y)$.
$\forall x \in S,\ P(x)$	for all x in S, $P(x)$ holds	Let an $x \in S$ be given. Prove $P(x)$, given that x is in the set S.	Choose some $y \in S$. We have $P(y)$.

my plan for the talk

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* pedagogical idea
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* evaluation

what worked

- nearly same exam in CS054, CS055
  nearly same mean score

- too hard and too much material
  students still had fun

	CS055	CS054	
TOP	20%	25%	can do all of the work
MID	60%	55%	can do most but not all work
COQ		15%	may not ever get a paper proof correct
BOT	20%	5%	may not ever get a proof correct

what didn't work

* CoqIDE was a nightmare
  * crashy, non-native UI
  * silently mangling Unicode characters
  * bad defaults for .vo files... needed CLI
* no documentation at an appropriate level
* grading informal proofs was awkward
* new material was rough
* no time to try graphs formally

challenges

* set theory **desiderata**:
  * **executable operations** on any type in Set
  * potentially **infinite**
  * allow for a treatment of **countability**
* less **arithmetic drudgery**
* more interesting **total programs**

sets

- axiomatized naïve typed set theory
  set : Type -> Type
  Universe : forall {X}, set X
- students did a bunch of equational proofs
  e.g., De Morgan's laws
- countability just on Coq's types
  formal proofs included:
  |nat| = |list unit| = |option nat|
  |nat| ≤ |nat -> nat|
  |x : set T| ≤ |power_set(x) : set (set T)|
  informal proofs included: $|\mathbb{N}| \leq |\mathbb{Q}|$, etc.

graphs

* wrote up inductive graphs following **Jean Duprat's GraphBasics**

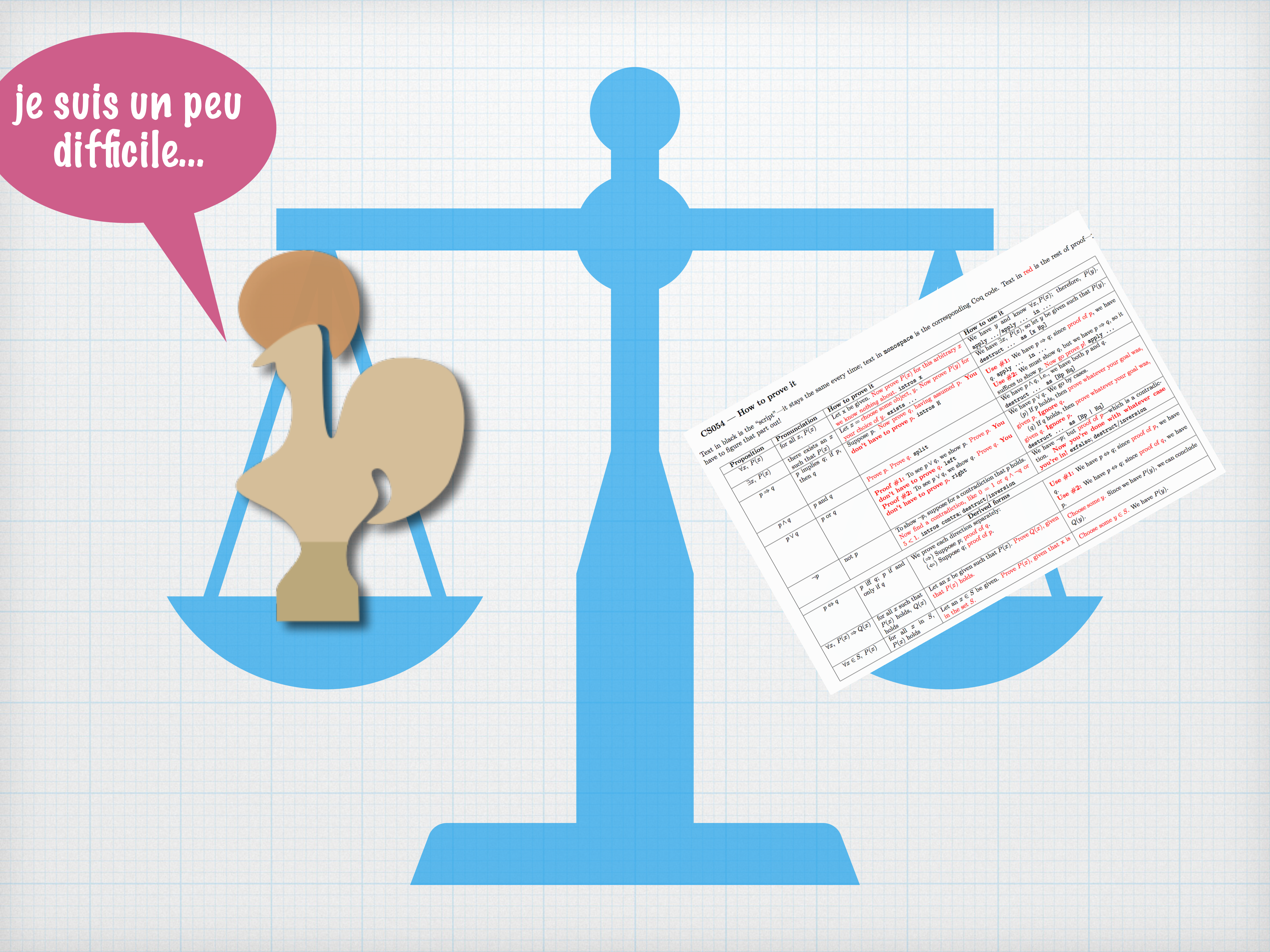
```
Inductive graph : list X -> list (X * X) -> Type :=
  | g_empty : graph [] []
  | g_vertex :
   forall (V : list X) (E : list (X * X))
          (g : graph V E) (v : X),
   ~In v V -> graph (v::V) E
  | g_arc :
   forall (V : list X) (E : list (X * X))
          (g : graph V E) (src tgt : X),
   In src V -> In tgt V -> ~ In (src,tgt) E ->
   graph V ((src,tgt)::E).
```

* proving **Euler's Handshaking Lemma** is easy: $\sum \deg(v) = 2 \cdot |E|$

  ...much harder with **standard maps and sums** on V and E!

what's next

* use **emacs**, spend a day on it and the CLI
* formal/informal as **separate submissions**
  * ...but keep the questions **related**!
* uniform treatment of **sums**
* **more programming**


je suis un peu difficile...
CS054 — How to prove it